

Mock exam 2: full solutions

Planetary Systems final exam practice • closed book • 120 minutes

Planetary Systems (WBAS002-05) • Kapteyn Astronomical Institute, University of Groningen

This mock exam has the same shape and level as the final exam: 6 questions of 10 points each, 60 points in total, in a 120-minute closed-book sitting where a calculator and a pen are all that is allowed. The twelve relations on the exam formula list are assumed known; every other formula a question needs is printed in the question itself. Marks per part are shown in brackets, and the steps carry marks, not only the number: show your algebra. Some parts state an intermediate result ("show that..."); use it to check your work, and to continue if a step resists. Carry at least four significant figures through the intermediate steps and round only at the end.

Constants and data

$G = 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ $1 \text{ AU} = 1.496 \times 10^{11} \text{ m}$ $1 \text{ yr} = 3.156 \times 10^7 \text{ s} = 365.25 \text{ d}$
 $1 \text{ d} = 86400 \text{ s}$ $M_{\odot} = 1.989 \times 10^{30} \text{ kg}$ $k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$ $m_u = 1.661 \times 10^{-27} \text{ kg}$
 $\sigma = 5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$ $S_{\odot} = 1361 \text{ W m}^{-2}$ (solar constant at 1 AU)

Body	a , AU	Further data
Venus	0.723 AU	
Earth	1.000 AU	$R = 6371 \text{ km}$, $\bar{\rho} = 5514 \text{ kg m}^{-3}$
Mars	1.524 AU	$M = 6.417 \times 10^{23} \text{ kg}$, $R = 3390 \text{ km}$
Saturn	9.537 AU, $A = 0.34$	$M = 5.683 \times 10^{26} \text{ kg}$, $R = 58232 \text{ km}$, effective temperature 95 K, outer edge of the main rings at 136800 km from the centre
Titan		$M = 1.345 \times 10^{23} \text{ kg}$, $R = 2575 \text{ km}$, upper-atmosphere $T = 150 \text{ K}$
Moon		$M = 7.342 \times 10^{22} \text{ kg}$, $R = 1737 \text{ km}$, dayside surface $T = 390 \text{ K}$
Comet	$q = 0.59 \text{ AU}$, $Q = 35.1 \text{ AU}$	

A total power L spread over a sphere of radius r gives a flux $L/(4\pi r^2)$; for sunlight this means $S = S_{\odot}/a^2$ with a in AU.

Problem 1 The orbit of a comet

A newly discovered comet orbits the Sun with perihelion distance q and aphelion distance Q as listed in the data table. For an ellipse with semi-major axis a and eccentricity e , the perihelion and aphelion distances are

$$q = a(1 - e), \quad Q = a(1 + e).$$

The vis-viva equation gives the speed at distance r on an orbit with semi-major axis a :

$$v^2 = GM_{\odot} \left(\frac{2}{r} - \frac{1}{a} \right),$$

and at the two extremes of the orbit, where the velocity is perpendicular to the radius, conservation of angular momentum gives $v_p q = v_a Q$.

(a) [3 pts] Compute the semi-major axis, the eccentricity, and the orbital period of the comet.

Solution. The major axis spans perihelion to aphelion:

$$a = \frac{q + Q}{2} = \frac{0.59 + 35.1}{2} \text{ AU} = 17.845 \text{ AU}.$$

Subtracting the two distance relations, $Q - q = 2ae$:

$$e = \frac{Q - q}{Q + q} = \frac{34.51}{35.69} = 0.9669.$$

Kepler's third law in ratio to Earth's orbit (P in years, a in AU):

$$P = a^{3/2} = 17.845^{3/2} = 75.38 \text{ yr}.$$

| $a = 17.85 \text{ AU}$, $e = 0.9669$, $P = 75.4 \text{ yr}$: **a Halley-type orbit.**

(b) [4 pts] Show that the comet's speed at perihelion is 54.4 km s^{-1} , then compute its speed at aphelion.

Solution. Vis-viva at $r = q = 0.59 \text{ AU} = 8.8264 \times 10^{10} \text{ m}$ with $a = 17.845 \text{ AU} = 2.6696 \times 10^{12} \text{ m}$ and $GM_\odot = 1.3275 \times 10^{20} \text{ m}^3 \text{ s}^{-2}$:

$$\begin{aligned} v_p^2 &= 1.3275 \times 10^{20} \left(\frac{2}{8.8264 \times 10^{10}} - \frac{1}{2.6696 \times 10^{12}} \right) \\ &= 1.3275 \times 10^{20} \times 2.2285 \times 10^{-11} = 2.9583 \times 10^9 \text{ m}^2 \text{ s}^{-2}, \end{aligned}$$

so $v_p = 54.39 \text{ km s}^{-1}$. Angular momentum conservation between the extremes:

$$v_a = v_p \frac{q}{Q} = 54.39 \times \frac{0.59}{35.1} = 0.914 \text{ km s}^{-1}.$$

| $v_p = 54.4 \text{ km s}^{-1}$, $v_a = 0.914 \text{ km s}^{-1}$: **a factor of 60 between the two ends of the orbit.**

(c) [3 pts] Judge this claim in 2 to 3 sentences, using your result from part (b): "A comet on this orbit spends most of its time in the inner solar system, where it is bright and easy to observe."

Solution. False. By Kepler's second law the orbital speed near aphelion (0.914 km s^{-1}) is 60 times smaller than near perihelion (54.4 km s^{-1}), so the comet crawls through the outer part of its orbit and races through the inner part. It spends the large majority of its 75-year period far from the Sun, cold and inactive; the bright phase near perihelion lasts only months.

| False: the comet moves slowest where it is farthest, so it spends almost all its time in the cold outer solar system.

Problem 2 The energy of building Mars

Consider the assembly of Mars from planetesimals. The gravitational binding energy of a uniform sphere of mass M and radius R ,

$$E = \frac{3}{5} \frac{GM^2}{R},$$

is the energy released when the body is assembled from material dispersed to infinity. The heat needed to warm a mass M by ΔT is $E = Mc_p \Delta T$; for rock, $c_p \approx 1200 \text{ J kg}^{-1} \text{ K}^{-1}$, and silicate rock starts to melt near 1400 K.

(a) [3 pts] Show that the accretion of Mars released about $4.9 \times 10^{30} \text{ J}$.

Solution. With the mass and radius from the data table:

$$E = \frac{3}{5} \frac{GM^2}{R} = \frac{3 \times 6.674 \times 10^{-11} \times (6.417 \times 10^{23})^2}{5 \times 3.390 \times 10^6} = \frac{8.2446 \times 10^{37}}{1.695 \times 10^7} = 4.864 \times 10^{30} \text{ J}.$$

| $E = 4.864 \times 10^{30} \text{ J}.$

(b) [3 pts] Assume for the moment that all of this energy is retained as heat. Compute the temperature rise of Mars and compare it with the melting temperature of rock. Explain in 1 to 2 sentences why the actual temperature rise was smaller, and what the comparison nevertheless implies for the early state of Mars.

Solution.

$$\Delta T = \frac{E}{Mc_p} = \frac{4.864 \times 10^{30}}{6.417 \times 10^{23} \times 1200} = 6317 \text{ K},$$

several times the melting temperature of rock. The actual rise was smaller because much of the accretion energy was radiated back to space as the surface was heated impact by impact; the estimate is an upper limit. The margin is so large, however, that even partial retention (deep energy burial by large impacts, plus the blanketing of an early atmosphere) melts a substantial part of the planet: early Mars went through a magma ocean stage.

| $\Delta T \approx 6300 \text{ K}$ if fully retained, far above the $\approx 1400 \text{ K}$ melting point: accretion energy is ample to melt Mars even if only a fraction is kept.

(c) [4 pts] Judge this claim in 3 to 4 sentences: “A planet without a global magnetic field today cannot have an iron core.”

Solution. False. A dynamo requires more than the presence of iron: it needs an electrically conducting fluid region stirred by convection. Mars is the counterexample: geodesy and seismology (its tidal response, and marsquake data) show a liquid iron core of about 1650 km radius, yet Mars has no global field today. Strongly magnetised ancient crust shows that Mars once ran a dynamo; as the small planet cooled, core convection became too sluggish to sustain it, and the field died while the core remained.

| False: a dynamo needs core convection, not just an iron core. Mars has the core but lost the convection, and with it the field.

Problem 3 Inside the Earth

Earth's radius and mean density are listed in the data table. For this problem, model the interior as two uniform layers: an iron core of density $\rho_c = 11\,000\text{ kg m}^{-3}$ and a rocky mantle of density $\rho_m = 4500\text{ kg m}^{-3}$ (both values already include compression). The mean density of such a body is the volume-weighted average of the two layer densities,

$$\bar{\rho} = \rho_c x^3 + \rho_m (1 - x^3), \quad x = \frac{R_c}{R},$$

where R_c is the core radius.

(a) [4 pts] Show that the two-layer model puts the core radius near 3400 km, then compute the fraction of Earth's mass that resides in the core.

Solution. Let $x = R_c/R$. The mean density is the volume-weighted average of the two layers:

$$\bar{\rho} = \rho_c x^3 + \rho_m (1 - x^3) \implies x^3 = \frac{\bar{\rho} - \rho_m}{\rho_c - \rho_m} = \frac{5514 - 4500}{11\,000 - 4500} = \frac{1014}{6500} = 0.1560.$$

So $x = 0.1560^{1/3} = 0.5383$ and

$$R_c = 0.5383 \times 6371\text{ km} = 3430\text{ km},$$

close to the seismologically measured 3480 km. The core mass fraction is

$$\frac{M_c}{M} = \frac{\rho_c x^3}{\bar{\rho}} = \frac{11\,000 \times 0.1560}{5514} = 0.311.$$

| $R_c \approx 3430\text{ km}$ (**measured:** 3480 km); **the core holds** $\approx 31\%$ **of Earth's mass** **in** $\approx 16\%$ **of its volume.**

(b) [3 pts] Inside a uniform sphere of density ρ , the local gravitational acceleration is

$$g(r) = \frac{4}{3}\pi G \rho r.$$

Integrate the equation of hydrostatic equilibrium from the surface to the centre to show that the central pressure of a uniform sphere is

$$P_c = \frac{2}{3}\pi G \bar{\rho}^2 R^2,$$

and evaluate it for Earth. The measured central pressure is 364 GPa; explain in 1 sentence why the uniform estimate falls short.

Solution. Hydrostatic equilibrium with the interior gravity of a uniform sphere:

$$\frac{dP}{dr} = -\rho g(r) = -\frac{4}{3}\pi G \rho^2 r \implies P(r) = \frac{2}{3}\pi G \rho^2 (R^2 - r^2),$$

integrating from the surface ($P = 0$) inward. At the centre:

$$P_c = \frac{2}{3}\pi G \bar{\rho}^2 R^2 = \frac{2}{3}\pi \times 6.674 \times 10^{-11} \times 5514^2 \times (6.371 \times 10^6)^2 = 1.725 \times 10^{11}\text{ Pa} = 172.5\text{ GPa}.$$

The estimate falls short of the measured 364 GPa because the real Earth concentrates

its mass in a dense core, which raises both the interior gravity and the pressure above the uniform-density values.

| $P_c \approx 170$ GPa, **about half the true value: central mass concentration raises the real pressure.**

(c) [3 pts] Judge this claim in 2 to 3 sentences: “Earth’s magnetic field is generated in the solid inner core.”

Solution. False. A dynamo needs organised motion of a conducting fluid, and a solid cannot convect: the field is generated by convection (thermal and compositional) in the *liquid outer core*, organised by Earth’s rotation. The solid inner core matters indirectly: its slow growth releases latent heat and expels light elements, and both help drive the outer-core convection.

| False: the dynamo runs in the convecting liquid outer core; the solid inner core only helps power it.

Problem 4 The rings and glow of Saturn

Saturn’s properties are listed in the data table. A strengthless (fluid) satellite of density ρ_m is torn apart by tides inside the Roche limit

$$d = 2.46 R_p \left(\frac{\rho_p}{\rho_m} \right)^{1/3},$$

where R_p and ρ_p are the radius and mean density of the planet.

(a) [3 pts] Compute Saturn’s mean density and compare it with the density of liquid water.

Solution.

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{5.683 \times 10^{26}}{\frac{4}{3}\pi(5.8232 \times 10^7)^3} = \frac{5.683 \times 10^{26}}{8.2712 \times 10^{23}} = 687.1 \text{ kg m}^{-3}.$$

| $\bar{\rho} = 687 \text{ kg m}^{-3}$, **less than water (1000 kg m⁻³): Saturn is the least dense planet in the solar system.**

(b) [4 pts] Take the ring particles to be icy material with $\rho_m = 1000 \text{ kg m}^{-3}$ as a round value. Show that the Roche limit lies near 126 000 km from Saturn’s centre. Compare it with the outer edge of the main rings and with the fact that all of Saturn’s sizeable moons orbit beyond 180 000 km, and state what this comparison says about why the rings exist.

Solution.

$$d = 2.46 \times 5.8232 \times 10^7 \times \left(\frac{687.1}{1000} \right)^{1/3} = 2.46 \times 5.8232 \times 10^7 \times 0.8823 = 1.264 \times 10^8 \text{ m} = 126\,400 \text{ km},$$

or about 2.2 Saturn radii. The main rings end at 136 800 km, about 8% beyond this nominal limit (real ring particles are porous, with densities well below that of solid ice, so the true limit lies farther out), while the sizeable moons all orbit well outside it. Inside the Roche limit tidal forces defeat self-gravity: ice fragments there cannot

accrete into a moon, and a moon that strays inside is pulled apart. The rings occupy exactly the zone where moons cannot exist. (The numerical coefficient depends on the satellite's strength and shape, so the limit marks a zone rather than a sharp line; the sharpness of the outer edge itself is maintained by a 7:6 orbital resonance with the small moon Janus, not by tidal physics alone.)

| $d \approx 126\,000\text{ km} \approx 2.2 R_{\text{Saturn}}$: **rings inside, moons outside. Tides prevent accretion inside the limit.**

(c) [3 pts] Compute the sunlight flux Saturn absorbs per square metre of its surface. Then use the measured effective temperature (data table) to show that Saturn emits about 1.9 times the power it absorbs, and explain in 1 to 2 sentences where the excess comes from.

Solution. The solar flux at Saturn is $S = S_{\odot}/a^2 = 1361/9.537^2 = 14.96\text{ W m}^{-2}$. Absorbed per unit surface area, with $A = 0.34$:

$$\frac{S}{4}(1 - A) = \frac{14.96}{4} \times 0.66 = 2.468\text{ W m}^{-2}.$$

The measured effective temperature of 95 K corresponds to an emitted flux

$$\sigma T_{\text{eff}}^4 = 5.670 \times 10^{-8} \times 95^4 = 4.618\text{ W m}^{-2},$$

a factor $4.618/2.468 = 1.87$ above the absorbed flux. The excess ($\approx 2.1\text{ W m}^{-2}$) is internal heat: slow gravitational contraction, and for Saturn in particular the settling of helium droplets through the metallic hydrogen interior (helium rain), both convert gravitational energy into heat.

| Absorbed 2.47 W m^{-2} versus emitted 4.62 W m^{-2} : Saturn radiates 1.9 times what it absorbs, powered by contraction and helium rain.

Problem 5 Keeping an atmosphere

Whether a body keeps an atmosphere depends on the race between its gravity and the thermal motion of gas molecules. A molecule of mass m in a gas at temperature T has the most probable speed

$$v_{\text{th}} = \sqrt{\frac{2k_{\text{B}}T}{m}},$$

and a useful rule of thumb is that a gas survives for billions of years only if the escape velocity exceeds about $6 v_{\text{th}}$ at the temperature of the upper atmosphere. Molecular masses: $m = \mu m_{\text{u}}$ with $\mu = 28$ for N_2 and $\mu = 2$ for H_2 .

(a) [3 pts] Compute the escape velocity from Titan's surface.

Solution.

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2 \times 6.674 \times 10^{-11} \times 1.345 \times 10^{23}}{2.575 \times 10^6}} = \sqrt{6.9721 \times 10^6} = 2640\text{ m s}^{-1}.$$

| $v_{\text{esc}} = 2.64\text{ km s}^{-1}$.

(b) [4 pts] Compute the most probable thermal speeds of N_2 and H_2 at the temperature of Titan's upper atmosphere, and apply the rule of thumb to each gas. What does the result explain about Titan's atmosphere?

Solution. For N_2 , $m = 28 \times 1.661 \times 10^{-27} = 4.6508 \times 10^{-26}$ kg:

$$v_{th} = \sqrt{\frac{2 \times 1.381 \times 10^{-23} \times 150}{4.6508 \times 10^{-26}}} = \sqrt{8.9082 \times 10^4} = 298 \text{ m s}^{-1},$$

so $v_{esc}/v_{th} = 2640/298.5 = 8.8 > 6$: nitrogen is retained. For H_2 , $m = 3.322 \times 10^{-27}$ kg:

$$v_{th} = \sqrt{\frac{4.1430 \times 10^{-21}}{3.322 \times 10^{-27}}} = \sqrt{1.2471 \times 10^6} = 1117 \text{ m s}^{-1},$$

so $v_{esc}/v_{th} = 2640/1117 = 2.4 < 6$: hydrogen escapes. This is why Titan keeps a dense N_2 (and methane) atmosphere while the hydrogen produced by methane photochemistry leaks away to space.

| $v_{th}(N_2) = 298 \text{ m s}^{-1}$ **(ratio 8.8: retained)**, $v_{th}(H_2) = 1117 \text{ m s}^{-1}$ **(ratio 2.4: lost)**.

(c) [3 pts] The Moon's escape velocity is close to Titan's, yet the Moon has no atmosphere. Compute the numbers that resolve this apparent contradiction.

Solution. The Moon's escape velocity is

$$v_{esc} = \sqrt{\frac{2 \times 6.674 \times 10^{-11} \times 7.342 \times 10^{22}}{1.737 \times 10^6}} = \sqrt{5.6420 \times 10^6} = 2375 \text{ m s}^{-1},$$

indeed nearly Titan's 2640 m s^{-1} . But the Moon orbits at 1 AU, and its dayside reaches 390 K: for N_2 ,

$$v_{th} = \sqrt{\frac{2 \times 1.381 \times 10^{-23} \times 390}{4.6508 \times 10^{-26}}} = \sqrt{2.3161 \times 10^5} = 481 \text{ m s}^{-1},$$

so $v_{esc}/v_{th} = 2375/481.3 = 4.9 < 6$: even nitrogen escapes over billions of years. Titan holds an atmosphere with the same gravity because at 9.5 AU it is cold; temperature, not just gravity, decides.

| Moon: $v_{esc} = 2.38 \text{ km s}^{-1}$ **but** $v_{th}(N_2) = 481 \text{ m s}^{-1}$ **at 390 K, ratio 4.9 < 6: too warm to keep gas that Titan, at the same gravity but 150 K, retains.**

Problem 6 The habitable zone

The habitable zone is the range of orbital distances in which a rocky planet with an Earth-like atmosphere can sustain liquid water on its surface. Climate models place it where the stellar flux S lies between about $0.35 S_{\oplus}$ (outer edge, maximum greenhouse) and $1.06 S_{\oplus}$ (inner edge, runaway greenhouse), with $S_{\oplus}$ the flux Earth receives. For a star of luminosity L , the flux at distance a (in AU) is

$$S = \frac{L}{L_{\odot}} \frac{S_{\oplus}}{a^2}.$$

(a) [3 pts] Compute the inner and outer edge of the Sun's habitable zone, in AU. Which of the planets of the solar system orbit inside it?

Solution. Setting $S/S_{\oplus} = 1/a^2$ equal to the two limits:

$$a_{\text{in}} = \sqrt{\frac{1}{1.06}} = 0.971 \text{ AU}, \quad a_{\text{out}} = \sqrt{\frac{1}{0.35}} = 1.690 \text{ AU}.$$

Earth (1.000 AU) and Mars (1.524 AU) orbit inside the zone; Venus (0.723 AU) lies well inside the inner edge.

| 0.971 to 1.690 AU: Earth and Mars are in the habitable zone, Venus is not.

(b) [4 pts] A red dwarf star has $L = 0.02 L_{\odot}$ and $M_{*} = 0.30 M_{\odot}$. Compute the edges of its habitable zone, and the orbital period of a planet at the inner edge.

Solution. The zone scales with $\sqrt{L/L_{\odot}}$:

$$a_{\text{in}} = \sqrt{\frac{0.02}{1.06}} = 0.1374 \text{ AU}, \quad a_{\text{out}} = \sqrt{\frac{0.02}{0.35}} = 0.239 \text{ AU}.$$

Kepler's third law in solar units (P in years, a in AU, M_{*} in solar masses), $P^2 = a^3 / (M_{*}/M_{\odot})$:

$$P = \sqrt{\frac{0.1374^3}{0.30}} = \sqrt{8.6473 \times 10^{-3}} = 0.09299 \text{ yr} = 34.0 \text{ d}.$$

| 0.137 to 0.239 AU; a planet at the inner edge orbits in 34.0 d. Habitable-zone planets of red dwarfs hug their star.

(c) [3 pts] Judge this claim in 2 to 3 sentences, using your result from part (a): "Mars orbits inside the Sun's habitable zone, so Mars is habitable today."

Solution. The premise is right but the conclusion is false: from part (a), Mars at 1.524 AU is indeed inside the habitable zone, yet its surface today is a frozen desert. The habitable zone only marks where an Earth-like atmosphere could sustain liquid water; Mars is too small to have held on to such an atmosphere, and its thin 6 mbar CO_2 envelope provides almost no greenhouse warming. Habitability needs both the right orbit and the right planet.

| False: the habitable zone is necessary in flux but not sufficient. Mars has the orbit and lacks the atmosphere.